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Asymptotic Behavior of Waves

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The asymptotic conjugation relation

$$\lim_{t \to \pm\infty} \| g(x/t)\, M(e^{it\rho}) f - M(g(\nabla\rho))\, M(e^{it\rho}) f \|_2 = 0 \qquad (*)$$

is established for all $f \in L^2(\mathbb{R}^n)$ under mild assumptions on ρ and g, where $M(h)f = \mathcal{F}^{-1}(h\hat{f})$ denotes Fourier multiplication. The asymptotic estimate

$$\lim_{t \to \pm\infty} \left\| \frac{x_j}{|x|} \frac{\partial u}{\partial t} \pm \frac{\partial u}{\partial x_j} \right\|_2 = 0$$

for finite energy solutions u of the wave equation is deduced from (*), along with generalizations to a class of first-order symmetric hyperbolic systems of partial differential equations that are homogeneous and constant coefficient, and a weakened version for the Klein–Gordon equation. Also deduced from (*) is the fact that for a free Schrödinger particle the probability of being in the set tA at time t tends to the probability that the velocity is in A as $t \to \pm\infty$.

> The wide wings flap but once to lift him up. A single ripple starts from where he stood.
>
> —from "The Heron" by Theodore Roethke [13]

1. INTRODUCTION

This paper is concerned with properties of functions $u(x, t) = \mathcal{F}^{-1}(e^{it\rho(\xi)} \hat{f}(\xi))$, where $\rho\colon \mathbb{R}^n \to \mathbb{R}^1$ and $f \in L^2(\mathbb{R}^n)$, and $\hat{f}(\xi) = \int f(x)\, e^{ix\cdot\xi}\, dx$ is the Fourier transform in $\mathbb{R}^n$. We shall refer to such functions as "waves," and this terminology is justified by the fact that different choices of the phase function ρ correspond to solutions of various wave equations of mathematical physics. The choice $\rho(\xi) = \pm c\,|\xi|$ gives solutions of the

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d'Alembertian wave equation $\partial^2 u/\partial t^2 = c^2 \Delta_x u$, the choice $\rho(\xi) = \pm\sqrt{m^2 + |\xi|^2}$ gives solutions to the Klein–Gordon equation $\partial^2 u/\partial t^2 = \Delta_x u - m^2 u$, while the choice $\rho(\xi) = c\,|\xi|^2$ gives solutions to the free Schrödinger equation $\partial u/\partial t = -ic\Delta_x u$.

By "asymptotic behavior" we refer to the limiting behavior a $t \to \pm\infty$, and in this paper we shall restrict our attention to L^2 limits, because these are the most physically interesting quantities and because it is unlikely that any of our results are valid pointwise or for L^p norms. Thus if $u(x, t)$ and $v(x, t)$ are two functions on space–time $\mathbb{R}^{n+1}$, not necessarily waves, we will write $u \sim v$ as $t \to \pm\infty$ to mean $\lim_{t \to \pm\infty} (\int_{\mathbb{R}^n} |u(x, t) - v(x, t)|^2\, dx)^{1/2} = 0$. Many results in scattering theory assert that solutions to more complicated equations than those mentioned above are asymptotically equal to these simpler waves, so the asymptotic behavior of waves discussed in this paper gives the same information about solutions of all equations that scatter in this sense.

Our main result is the asymptotic relation

$$g(x/t)\, u(x, t) \sim M(g(\nabla\rho))\, u(x, t) \tag{*}$$

as $t \to \pm\infty$ where we denote by $M(h)$ the operation of Fourier multiplication by h, $M(h)f = \mathcal{F}^{-1}(h \cdot \hat{f})$. The asymptotic conjugation relation (*) will be established in Section 2 under very mild assumptions on ρ and g. It is an elementary result in Fourier analysis, related to the familiar conjugation relation $e^{ix\cdot\lambda} f(x) = \mathcal{F}^{-1}(\hat{f}(\xi + \lambda))$, and yet it has very far-reaching applications, enabling us to unify and generalize many known results.

As a first application (Section 3), we show that the probability that a free quantum-mechanical particle at time t lies in the set tA tends as $t \to \pm\infty$ to the probability that its velocity lies in A (the velocity probability is time independent). In other words, in the long run the position statistics for a free quantum-mechanical particle tend to the position statistics for a random classical particle whose velocity is distributed according to the velocity statistics of the quantum-mechanical particle. While this result is well known [12, Vol. II, Theorem IX.31], our proof is quite simple. It would be interesting to extend this result to solutions to the Schrödinger equation with a potential which is too strong for complete scattering (some work in this direction has been done by Enss [8]).

Next we consider solutions of the d'Alembertian wave equation (for simplicity take $c = 1$) with finite energy, $u(x, t) = \mathcal{F}^{-1}(|\xi|^{-1} e^{it|\xi|} \hat{f}_+(\xi) + |\xi|^{-1} e^{-it|\xi|} \hat{f}_-(\xi))$ with $f_\pm \in L^2$. It was shown by Brodsky [4] and Lax and Phillips [10] that asymptotically the total energy $E(t) = \frac{1}{2}\int (|u_t(x, t)|^2 + |\nabla_x u(x, t)|^2)\, dx$, which is time independent, divides equally into kinetic $\frac{1}{2}\int |u_t(x, t)|^2\, dx$ and potential $\frac{1}{2}\int |\nabla_x u(x, t)|^2\, dx$ parts. This equipartition theorem was recently further refined by Glassey and Strauss [9] who showed that asymptotically the space–time gradient $(u_t, \nabla_x u)$ lies in the one-dimensional space spanned by $(\mp |x|, x)$ as $t \to \pm\infty$ (more precisely,

$$\frac{\partial u}{\partial x_j} \sim \mp \frac{x_j}{|x|} \frac{\partial u}{\partial t}$$

as $t \to \pm\infty$). We will show (in Section 4) that this is a simple consequence of the asymptotic conjugation relation. We can interpret this result to say that the wave asymptotically resembles an outgoing or incoming "ripple" $f(|x| \mp t)$. Thus the "single ripple" in Roethke's poem quoted above, which cannot accurately describe reality in the short run because of the failure of Huyghen's principle in even dimensions, turns out to be a rather apt description over a longer time scale.

For solutions of the Klein–Gordon equation with finite energy the exact analogy of the above theorem is false, but there is a weaker version, namely, the gradient asymptotically is localized to the one-dimensional space spanned by $(-t, x)$. There is also a further asymptotic relation between u and its gradient that holds only for positive energy solutions, $iu \sim m^{-1}\sqrt{1 - |x|^2/t^2}\,(\partial u/\partial t)$ as $t \to \pm\infty$, while for negative energy solutions the same relation holds with a minus sign.

Our last application (Section 5) is to symmetric first-order hyperbolic systems of partial differential equations which are both constant coefficient and homogeneous. Associated with such equations are bicharacteristic flows, and a celebrated theorem of Courant and Lax [6] asserts that all singularities of the solution propagate along the bicharacteristic flows (this result does not require the hypotheses that the equations have constant coefficients and be homogeneous). We will show for a large class of such equations, which we have dubbed *well-rounded*, that asymptotically all the energy is propagated along the bicharacteristic flows. This is the exact analogue of the wave equation theorem and it generalizes results of Costa and Strauss [5], and it is again a simple consequence of the asymptotic conjugation relation. It would be of interest to extend this result to the variable coefficient case; however the standard treatments of variable coefficient hyperbolic equations using Fourier integral operators (see [7]) do not seem adequate because they are only accurate locally.

2. The Asymptotic Conjugation Relation

Let $\rho(\xi)$ be a measurable real-valued function on $\mathbb{R}^n$. Then $M(e^{it\rho})$ is a group of unitary operators on L^2 for $t \in \mathbb{R}$. If $g \in L^\infty(\mathbb{R}^n)$, then multiplication by $g(x/t)$ is a bounded operator on L^2 for $t \neq 0$. The asymptotic conjugation relation says that $M(e^{it\rho})^{-1} g(x/t) M(e^{it\rho})$ converges in the strong operator topology to $M(g(\nabla\rho(\xi))$ as $t\to\infty$ under very mild assumptions on ρ and g. To begin with we need only assume that ρ is differentiable a.e., and g is a Fourier–Stieltjes transform. This means $\nabla\rho$ is defined a.e. and measurable, so $g(\nabla\rho(\xi))$ is a well-defined L^∞ function.

We can rewrite the asymptotic conjugation relation as

$$\| g(x/t)\, M(e^{it\rho})f - M(g(\nabla\rho))\, M(e^{it\rho})f\|_2 \to 0 \qquad \text{as} \quad t\to\infty$$
$$\text{for all} \quad f\in L^2. \qquad (*)$$

It is clear by elementary functional analysis that if suffices to prove this for f in a dense subspace of L^2, and also the class of functions g for which it holds is closed under uniform limits. We will make frequent use of these observations.

THEOREM 2.1. *Let $\rho\colon \mathbb{R}^n\to\mathbb{R}$ be differentiable almost everywhere. Then the asymptotic conjugation relation* (*) *holds for every g which is a uniform limit of Fourier–Stieltjes transforms.*

Proof. The idea of the proof is quite simple. Suppose g were of the form $e^{ix\cdot\lambda}$ for some $\lambda\in\mathbb{R}^n$. Then $e^{ix\cdot\lambda/t}M(e^{it\rho})f=\mathscr{F}^{-1}(e^{it\rho(\xi+\lambda/t)}\hat{f}(\xi+\lambda/t))$. If we let $t\to\infty$, then $\hat{f}(\xi+\lambda/t)$ converges to $\hat{f}(\xi)$ in L^2 norm, and $e^{it\rho(\xi+\lambda/t)}$ converges to $e^{i\nabla\rho(\xi)\cdot\lambda}e^{it\rho(\xi)}$ at the points ξ where ρ is differentiable. Thus, at least formally, $\mathscr{F}^{-1}(e^{it\rho(\xi+\lambda/t)}\hat{f}(\xi+\lambda/t))$ converges to $\mathscr{F}^{-1}(e^{i\nabla\rho(\xi)\cdot\lambda}e^{it\rho(\xi)}\hat{f}(\xi))$, and it only remains to verify that the convergence takes place in the L^2 norm. More generally, if g is a Fourier–Stieltjes transform, $g(x)=\int e^{ix\cdot\lambda}\,d\mu(\lambda)$, for a finite measure μ, it is merely a question of integrating both sides with respect to $d\mu(\lambda)$.

The technical details of the proof require the use of the dominated convergence theorem, and to facilitate this we assume first that μ has bounded support and $\hat{f}(\xi)$ is continuous and has bounded support. These restrictions are easily removed by passing to the limit as indicated above. Under these assumptions there exists a function $h(\xi)\in L^1\cap L^2$ such that $|\hat{f}(\xi+\lambda/t)|\leqslant h(\xi)$ for all $|t|\geqslant 1$ and all λ in the support of μ. Let $H(t,\xi,\lambda)=e^{it\rho(\xi+\lambda/t)}\hat{f}(\xi+\lambda/t)-e^{it\rho(\xi)}e^{i\lambda\cdot\nabla\rho(\xi)}\hat{f}(\xi)$. Then $H(t,\xi,\lambda)\to 0$ as $t\to\infty$ for every λ in the support of μ and for each ξ for which ρ is differentiable. Also $|H(t,\xi,\lambda)|\leqslant 2h(\xi)$ for $|t|\geqslant 1$; so by the dominated convergence theorem $\int H(t,\xi,\lambda)\,d\mu(\lambda)\to 0$ as $t\to\infty$ for such points ξ.

If we let $F(t,\xi)=\int H(t,\xi,\lambda)\,d\mu(\lambda)$, we have shown $F(t,\xi)\to 0$ as $t\to\infty$ for almost every ξ. But

$$|F(t,\xi)|\leqslant\int |H(t,\xi,\lambda)|\,d|\mu|(\lambda)\leqslant 2h(\xi)\,\|\mu\|$$

for $|t| \geqslant 1$ so by the L^2 dominated convergence theorem $F(t, \xi) \to 0$ in L^2. But in fact $\mathscr{F}^{-1} F(t, \xi) = g(x/t)\, M(e^{it\rho}) f - M(g(\nabla\rho))\, M(e^{it\rho}) f$ where $g(x) = \int e^{ix \cdot \lambda}\, d\mu(\lambda)$, since the λ and ξ integrals may be interchanged by Fubini's theorem. Thus (*) follows by the Plancherel theorem. Q.E.D.

For many applications it is necessary to have the asymptotic conjugation relation hold for discontinuous functions g. Thus we need to extend the theorem to allow more general functions g, and this will require more assumptions on ρ. We first show that (*) holds for all $g \in L^\infty$ under rather stringent restrictions on ρ, which are not satisfied by all of the examples. Let ν denote the pull-back of Lebesgue measure via the function $\nabla\rho$, $\nu(A) = |\{\xi : \nabla\rho(\xi) \in A\}|$ where $|\ |$ denotes Lebesgue measure. We will assume ν is absolutely continuous with respect to Lebesgue measure. This condition is satisfied by $\rho(\xi) = c\,|\xi|^2$, but not by $\rho(\xi) = |\xi|$, for example.

COROLLARY 2.2. *Assume in addition that ν is absolutely continuous with respect to Lebesgue measure. Then the asymptotic conjugation relation* (*) *holds for all $g \in L^\infty$.*

Proof. The idea of the proof is that (*) is preserved under limits slightly weaker than uniform in g. To simplify notation write $u(t) = M(e^{it\rho}) f$. If g_n is a *uniformly bounded* sequence of measurable functions for which (*) is known to hold, and if g is $\lim_{n \to \infty} g_n$ in some weak sense, we will estimate

$$\begin{aligned} &\| g(x/t)\, u(t) - M(g(\nabla\rho))\, u(t)\|_2 \\ &\quad \leqslant \| g(x/t)\, u(t) - g_n(x/t)\, u(t)\|_2 \\ &\qquad + \| g_n(x/t)\, u(t) - M(g_n(\nabla\rho))\, u(t)\|_2 \\ &\qquad + \| M(g_n(\nabla\rho))\, u(t) - M(g(\nabla\rho))\, u(t)\|_2 . \end{aligned}$$

We need to show this can be made arbitrarily small by taking $|t|$ large enough, in order to have (*) hold for g. We will do this as follows: given $\varepsilon > 0$, choose n large enough such that the first and third terms are less than $\varepsilon/3$ for all large $|t|$; then with n fixed use (*) for g_n to conclude the second term is less than $\varepsilon/3$ for $|t|$ large enough.

Now the third term presents no real difficulty since

$$\begin{aligned} &\| M(g_n(\nabla\rho))\, u(t) - M(g(\nabla\rho))\, u(t)\|_2 \\ &\quad = \|(g_n(\nabla\rho(\xi)) - g(\nabla\rho(\xi)))\, e^{it\rho(\xi)} \hat{f}(\xi)\|_2 \\ &\quad = \|(g_n(\nabla\rho(\xi)) - g(\nabla\rho(\xi)))\, \hat{f}(\xi)\|_2 \end{aligned}$$

which is independent of t, and goes to zero as $n \to \infty$ by the dominated convergence theorem as long as $g_n(\nabla\rho(\xi)) \to g(\nabla\rho(\xi))$ a.e. since g_n is uniformly bounded. But if we require $g_n(x) \to g(x)$ a.e. we will have $g_n(\nabla\rho(\xi)) \to g(\nabla\rho(\xi))$ a.e. because ν is absolutely continuous.

The first term is more delicate and requires a new idea. Suppose h_k is a uniformly bounded sequence of functions for which (*) holds, say h_k are Fourier–Stieltjes transforms, and such that $h_k(x) \to 0$ a.e. Then $h_k(\nabla\rho(\xi)) \to 0$ a.e.; so by (*) and the dominated convergence theorem, given any $\varepsilon > 0$ there exists k such that $\| h_k(x/t)\, u(t)\|_2 \leqslant \varepsilon$ for all sufficiently large t. Also, if D_k is a measurable set and $\chi_{D_k} \leqslant h_k$, then $\|\chi_{D_k}(x/t)\, u(t)\|_2 \leqslant \varepsilon$ for t large. Finally if $g_n \to g$ uniformly on the complement D_k' of D_k, then

$$\|(g(x/t) - g_n(x/t))\, u(t)\|_2 \leqslant \|(g(x/t) - g_n(x/t))\, \chi_{D_k}(x/t)\, u(t)\|_2 + \|(g(x/t) - g_n(x/t))\, \chi_{D_k'}(x/t)\, u(t)\|_2 .$$

By taking k large the first term is dominated by $2c\varepsilon$ where c is a uniform bound for $g_n(x)$, and then by taking n large enough the second term can be dominated by ε since $g - g_n$ is small on D_k'.

Altogether then, we have shown that the class of functions g for which (*) holds is closed under uniformly bounded limits which are uniform on a sequence of sets D_k' such that there exist Fourier–Stieltjes transforms h_k with $\chi_{D_k} \leqslant h_k$ and $h_k \to 0$ a.e. This allows us to deduce (*) if $g = \chi_A$ where A is a bounded measurable set. For by regularity of Lebesgue measure there exist compact sets F_k and open sets G_k such that $F_k \subset A \subset G_k$ and $D_k = G_k \backslash F_k$ has Lebesgue measure less than $1/k$. We can also take G_k bounded and decreasing and F_k increasing. Then there exist Fourier–Stieltjes transforms g_k with support in G_k and identically one on F_k. Notice that $g_n = g$ on D_k' once $n \geqslant k$ so $g_n \to g$ uniformly on D_k'. Also D_k are bounded open decreasing sets whose measure tends to zero, so there is no difficulty in finding Fourier–Stieltjes transforms h_k with $h_k \geqslant \chi_{D_k}$ and $h_k \to 0$ on the complement of $\bigcap D_k$. Thus (*) holds for $g = \chi_A$.

Next assume A is an arbitrary measurable set. Then if g_n denotes the characteristic function of $A \cap \{|x| \leqslant n\}$, (*) holds for g_n by the above, and $g_n \to \chi_A$ uniformly on the complement of $D_k = \{|x| > k\}$. It is easy enough to find Fourier–Stieltjes transforms h_k which are one on D_k and zero for $|x| \leqslant k/2$, so $h_k \to 0$ and $\chi_{D_k} \leqslant h_k$. Thus (*) holds for $g = \chi_A$.

From this it follows that (*) holds for g simple, and simple functions are uniformly dense in L^∞. Q.E.D.

Even if ν is not absolutely continuous, we can extend the asymptotic conjugation relation to functions g which are well behaved near the singular part of ν. The following is probably not the best possible result, but it will do for the applications.

COROLLARY 2.3. *Let $\nu = \nu_a + \nu_s$ be the Lebesgue decomposition into the absolutely continuous and singular part with respect to Lebesgue measure.*

Suppose the singular part ν_s has support in a bounded set S. Then the asymptotic conjugation relation (*) *holds for any $g \in L^\infty$ that is continuous in a neighborhood of S.*

Proof. Write $g = g_1 + g_2$ where g_1 is continuous with compact support and g_2 has support bounded away from S. Then the proof of the previous corollary establishes (*) for g_2, while the theorem gives (*) for g_1. Q.E.D.

We conclude this section with two consequences of the asymptotic conjugation relation concerning the asymptotic localization of the wave $u(t) = M(e^{it\rho})f$ in space.

THEOREM 2.4. *Let A be any measurable set containing the range of $\nabla\rho$. Assume either ν is absolutely continuous or that the singular part ν_s has bounded support and A contains a neighborhood of the support of ν_s. Then*

$$\|\chi_A(x/t)\, u(t) - u(t)\|_2 \to 0 \qquad \text{as} \quad t \to \pm\infty.$$

Proof. We choose $g = \chi_A$ in Corollary 2.2 or 2.3 (in this case g is continuous in a neighborhood of the support of ν_s). The theorem follows immediately from (*) since $g(\nabla\rho(\xi)) = 1$ a.e. Q.E.D.

THEOREM 2.5. *Assume* $\nabla\rho(\xi) \neq 0$ a.e. (*this is the same as saying* $\nu_s(\{0\}) = 0$). *Then if $h(t)$ is any monotonically increasing function tending to $+\infty$ as $t \to +\infty$, we have*

$$\|\chi(h(|t|)^{-1}\,|t| \leqslant |x| \leqslant h(|t|)\,|t|)\, u(t) - u(t)\|_2 \to 0 \qquad \text{as} \quad t \to \pm\infty.$$

Proof. Choose a sequence of Fourier–Stieltjes transforms g_n such that $g_n(x) = 1$ for $n^{-1} \leqslant |x| \leqslant n$, $g_n(x) = 0$ for $|x| \leqslant (2n)^{-1}$ or $|x| \geqslant 2n$, and $0 \leqslant g_n(x) \leqslant 1$ everywhere. Then $g_n(\nabla\rho(\xi)) \to 1$ as $n \to \infty$ if $\nabla\rho(\xi) \neq 0$, hence a.e., so $M(g_n(\nabla\rho))\, u(t) \to u(t)$ in L^2 norm as $n \to \infty$ uniformly in t by the dominated convergence theorem.

Given any $\varepsilon > 0$, choose n large enought that $\|M(g_n(\nabla\rho))\, u(t) - u(t)\|_2 \leqslant \varepsilon$ for all t. We then write

$$\chi_t u - u = (\chi_t u - u)\, g_n(x/t) + (\chi_t u - u)(1 - g_n(x/t)),$$

where $\chi_t(x) = \chi(h(|t|)^{-1}\,|t| \leqslant |x| \leqslant h(|t|)\,|t|)$. Once $|t|$ is large enough, $\chi_t(x)\, g_n(x/t) = g_n(x/t)$, so the first term vanishes. Thus

$$\|\chi_t u - u\|_2 = \|(\chi_t u - u)(1 - g_n(x/t))\|_2 \leqslant 2\, \|u - g_n(x/t)u\|_2$$

since $\chi_t(x) \leqslant 1$. But by (*) $\|g_n(x/t)u - M(g_n(\nabla\rho))u\|_2 \to 0$ as $t \to \pm\infty$ so $\lim_{t \to \pm\infty} \|\chi_t u - u\|_2 \leqslant 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves the result.

Q.E.D.

3. The Free Schrödinger Equation

The equation $\partial u/\partial t = ic\Delta_x u$, where $c = \hbar/2m$, describes the time evolution of a free quantum-mechanical particle of mass m. The solutions are the waves $u = M(e^{-ic|\xi|^2 t})f$ with $f \in L^2$, and it is convenient to normalize f to have $\|f\|_2 = 1$. Then $\int_A |u(x, t)|^2\, dx$ is interpreted as the probability that the particle lies in the set A at time t. The probability that the velocity lies in A (this is independent of time for free particles) is given by $(2\pi)^{-n} \int_{(-A/2c)} |\hat{f}(\xi)|^2\, d\xi$. Here we denote the dilation of a set A by the scalar λ as $\lambda A = \{\lambda x : x \in A\}$. For an earlier proof of the following theorem see [12, Vol. II, Theorem IX.31].

Theorem 3.1. *For any solution u of the free Schrödinger equation and any measurable set A we have*

$$\lim_{t \to \pm\infty} \int_{tA} |u(x, t)|^2\, dx = (2\pi)^{-n} \int_{(-A/2c)} |\hat{f}(\xi)|^2\, d\xi.$$

In other words, the probability that the particle lies in tA tends to the probability that the velocity lies in A.

Proof. We apply Corollary 2.2 with $\rho(\xi) = -c|\xi|^2$, so $\nabla\rho(\xi) = -2c\xi$, and $g(x) = \chi_A(x)$. The asymptotic conjugation relations says $\chi_A(x/t)u - M(\chi_A(-2c\xi))u$ tends to zero in L^2 norm as $t \to \pm\infty$. This implies $\|\chi_A(x/t)u\|_2^2$ and $\|M(\chi_A(-2c\xi))u\|_2^2$ have the same limit. Now $\|\chi_A(x/t)u\|_2^2 = \int_{tA} |u(x, t)|^2\, dx$. On the other hand

$$\|M(\chi_A(-2c\xi)u\|_2^2 = (2\pi)^{-n} \int_{(-A/2c)} |\hat{f}(\xi)|^2\, d\xi$$

by the Plancherel formula. Q.E.D.

4. The Wave Equation and Klein–Gordon Equation

Let u be a solution of the wave equation $\partial^2 u/\partial t^2 = \Delta_x u$ with finite energy $E(t) = \frac{1}{2} \int (|u_t|(x, t)|^2 + |\nabla_x(x, t)|^2)\, dx$. Then u is a superposition of two waves, $u = M(|\xi|^{-1} e^{it|\xi|})f_+ + M(|\xi|^{-1} e^{-it|\xi|})f_-$ with f_+ and f_- in L^2. The factor $|\xi|^{-1}$ is relatively harmless and disappears when we consider $\partial u/\partial t$. The phase functions are $\rho = \pm|\xi|$ and $\nabla\rho = \pm\xi/|\xi|$. Thus the range of $\nabla\rho$ is the unit sphere, and this is also the support of the singular measure ν.

We begin by showing that asymptotically all the energy is localized in space to the spherical shell $S_\varepsilon(t) = \{x : (1-\varepsilon)|t| \leqslant |x| \leqslant (1+\varepsilon)|t|\}$ for any fixed $\varepsilon > 0$. This result is well known, being an immediate consequence of

Huyghens' principle when n is odd, and can be thought of as a weak form of Huyghens' principle since it asserts that asymptotically all the energy propagates at a speed between $1-\varepsilon$ and $1+\varepsilon$.

THEOREM 4.1. *For any $\varepsilon > 0$ and any solution u of the wave equation with finite total energy, the energy in the complement $S_\varepsilon(t)'$ of the spherical shell*

$$\frac{1}{2}\int_{S_\varepsilon(t')} (|u_t(x,t)|^2 + |\nabla_x u(x,t)|^2)\,dx$$

tends to zero as $t \to \pm\infty$.

Proof. We apply Theorem 2.4 with $A = \{1-\varepsilon \leqslant |x| \leqslant 1+\varepsilon\}$ to the waves $u_t = iM(e^{it|\xi|})f_+ - iM(e^{-it|\xi|})f_-$ and

$$\frac{\partial u}{\partial x_j} = -iM(e^{it|\xi|})\,M\left(\frac{\xi_j}{|\xi|}\right)f_+ - iM(e^{-it|\xi|})\,M\left(\frac{\xi_j}{|\xi|}\right)f_-.$$

Note that $M(i(\xi_j/|\xi|))f_\pm$ are also in L^2. The conclusion

$$\|\chi_A(x/t)\,u_t - u_t\|_2 \to 0 \qquad \text{and} \qquad \left\|\chi_A(x/t)\frac{\partial u}{\partial x_j} - \frac{\partial u}{\partial x_j}\right\|_2 \to 0$$

as $t \to \pm\infty$ of Theorem 2.4 gives the desired result. Q.E.D.

Next we come to the theorem [9] that the energy vector grad $u = (u_t, \nabla_x u)$ is asymptotically localized to the one-dimensional space spanned by $(\mp|x|, x)$. This may be loosely interpreted to mean that the wave u asymptotically resembles a simple ripple $f(|x| \mp t)$, but we must refrain from reading too much into this statement. For instance, the ripple is radial, and while the contribution of all the tangential components of $\nabla_x u$ to the energy tends to zero, it is *not* true that u can be approximated in energy norm by radial functions.

THEOREM 4.2. *Let u be a finite solution of the wave equation. Then*

$$\left\|\frac{x_j}{|x|}\frac{\partial u}{\partial t} \pm \frac{\partial u}{\partial x_j}\right\|_2 \to 0 \qquad \textit{as} \quad t \to \pm\infty.$$

Proof. We apply Corollary 2.3 with $g(x) = x_j/|x|$ to the wave u_t. Note that g is continuous except at the origin, hence in a neighborhood of the support of v_s. Also g is odd and homogeneous of degree zero so $g(x/t) = g(x)$ if $t > 0$, $g(x/t) = -g(x)$ if $t < 0$. Then

$$g(x)\,u_t = ig(x/t)\,M(e^{it|\xi|})f_+ - ig(x/t)\,M(e^{-it|\xi|})f_- \qquad \text{for} \quad t > 0$$

is asymptotic as $t \to +\infty$ to

$$iM\left(\frac{\xi_j}{|\xi|}\right) M(e^{-it|\xi|})f_+ + iM\left(\frac{\xi_j}{|\xi|}\right) M(e^{it|\xi|})f_- = -\frac{\partial u}{\partial x_j}$$

which proves the result for $t \to +\infty$. A similar argument shows $g(x)\,u_t$ is asymptotic to $+\partial u/\partial x_j$ as $t \to -\infty$ since the only change is that now $g(x) = -g(x/t)$. Q.E.D.

We can combine the two results to say that grad u is asymptotically localized both in space (to the shell $S_\varepsilon(t)$) and in direction (to the span of $(\mp |x|, x)$). In particular, all the rotational derivatives

$$\frac{x_j}{|x|}\frac{\partial u}{\partial x_k} - \frac{x_k}{|x|}\frac{\partial u}{\partial x_j}$$

go to zero in L^2 norm, as does the incoming or outgoing derivative $\partial u/\partial t \pm \partial u/\partial r$, while all the energy is concentrated in the $\partial u/\partial t \mp \partial u/\partial r$ derivative.

We next examine the finite energy solutions of the Klein–Gordon equation $\partial^2 u/\partial t^2 = \Delta_x u - m^2 u$. These are again given as a superposition of waves $u = u_+ + u_-$, where

$$u_\pm = M(\exp(\pm it\sqrt{m^2 + |\xi|^2}))\, M((m^2 + |\xi|^2)^{-1/2})f_\pm$$

with $f_\pm \in L^2$. The waves u_+ and u_- correspond to positive and negative energy solutions, respectively, and in this case their asymptotic behavior is not identical. The phase functions are $\rho = \pm\sqrt{m^2 + |\xi|^2}$, so $\nabla\rho(\xi) = \pm\xi/\sqrt{m^2 + |\xi|^2}$. In this case the range of $\nabla\rho$ is the unit ball $\{|\xi| < 1\}$ and the measure ν is absolutely continuous. The energy now involves u as well as its gradient,

$$E(t) = \tfrac{1}{2}\int (|u_t(x,t)|^2 + |\nabla_x u(x,t)|^2 + m^2 |u(x,t)|^2)\, dx.$$

The localization given in Theorem 4.1 does not hold, but rather by applying Theorem 2.4 to the set $A = \{|x| < 1\}$ we obtain $\lim_{t \to \pm\infty} \frac{1}{2}\int_{|x| > |t|} (|u_t|^2 + |\nabla_x u|^2 + m^2 |u|^2)\, dx = 0$. Of course Theorem 2.5 also applies; so by combining the two we find that the energy is asymptotically localized in space to the spherical shell $\{x : h(|t|)^{-1}|t| \leqslant |x| \leqslant |t|\}$, where $h(t) \nearrow +\infty$ as $t \to \infty$. Actually the same method of proof yields a slightly sharper result: the spherical shells $\{x : \varepsilon(|t|)\,|t| \leqslant |x| \leqslant (1 - \varepsilon(|t|))\,|t|\}$ will do, where $\varepsilon(t) = h(t)^{-1}$. We can paraphrase this by saying that asymptotically all the energy propagates at speeds strictly between zero and one. This shows that the failure of Huyghen's principle for the Klein–Gordon equation is of a different order of magnitude from its failure for the wave equation in even space dimensions. These facts are also consequences of the $O(t^{-n/2})$ decay valid for a dense class of solutions.

The gradient of u can again be localized to a one-dimensional space, but now this space depends on t as well as x. Since the total energy involves u as well as its gradient, we expect also an asymptotic relation involving u and its gradient. Such a relation exists separately for the positive and negative energy solutions.

THEOREM 4.3. *Let $u = u_+ + u_-$ be any finite energy solution of the Klein–Gordon equation. Then*

$$\lim_{t \to \pm\infty} \left\| \frac{x_j}{t} \chi(|x| \leqslant |t|) \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_j} \right\|_2 = 0,$$

$$\lim_{t \to \pm\infty} \left\| \frac{1}{m} \sqrt{1 - \frac{|x|^2}{t^2}} \chi(|x| \leqslant |t|) \frac{\partial u_+}{\partial t} - iu_+ \right\|_2 = 0,$$

and

$$\lim_{t \to \pm\infty} \left\| \frac{1}{m} \sqrt{1 - \frac{|x|^2}{t^2}} \chi(|x| \leqslant |t|) \frac{\partial u_-}{\partial t} + iu_- \right\|_2 = 0.$$

Proof. We have

$$\frac{\partial u}{\partial t} = iM(\exp(it\sqrt{m^2 + |\xi|^2}))f_+ - iM(\exp(-it\sqrt{m^2 + |\xi|^2}))f_-$$

and

$$\frac{\partial u}{\partial x_j} = iM(\exp(it\sqrt{m^2 + |\xi|^2}))\, M\left(\frac{\xi_j}{\sqrt{m^2 + |\xi|^2}}\right) f_+$$
$$-iM(\exp(-it\sqrt{m^2 + |\xi|^2}))\, M\left(\frac{\xi_j}{\sqrt{m^2 + |\xi|^2}}\right) f_- .$$

To establish the first asymptotic relation we apply Corollary 2.2 with $g(x) = x_j \chi(|x| \leqslant 1)$ to the wave $\partial u / \partial t$. Then

$$g(x/t) \frac{\partial u}{\partial t} = \frac{x_j}{t} \chi(|x| \leqslant |t|) \frac{\partial u}{\partial t} \qquad \text{and} \qquad M(g(\nabla\rho)) \frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x_j}.$$

To establish the second and third asymptotic relation we use $g(x) = (1/m)\sqrt{1-|x|^2}\,\chi(|x| \leqslant 1)$. Then

$$g(x/t)\frac{\partial u_\pm}{\partial t} = \frac{1}{m}\sqrt{1-\frac{|x|^2}{t^2}}\,\chi(|x| \leqslant |t|)\frac{\partial u_\pm}{\partial t}$$

and

$$M(g(\nabla\rho))\frac{\partial u_\pm}{\partial t} = M((m^2+|\xi|^2)^{-1/2})\frac{\partial u_\pm}{\partial t} = \pm iu_\pm. \qquad \text{Q.E.D.}$$

In particular, we observe that the rotational derivatives

$$\frac{x_j}{|x|}\frac{\partial u}{\partial x_k} - \frac{x_k}{|x|}\frac{\partial u}{\partial x_j}$$

still tend to zero. Notice, too, that the one-dimensional space to which grad u is localized at the point (x, t) is spanned by the gradient of a ripple $f(|x| \mp \lambda t)$ travelling at speed $\lambda = |x|/|t|$, $0 \leqslant \lambda \leqslant 1$.

5. Homogeneous Symmetric Hyperbolic Systems

Let $u(x, t)$ denote now a function taking values in $\mathbf{C}^m$. A homogeneous, constant coefficient, first-order symmetric hyperbolic system of partial differential equations is a system of the form

$$\frac{\partial}{\partial t}u_i = \sum_{j=1}^{m}\sum_{k=1}^{n} A_{ijk}\frac{\partial u_j}{\partial x_k}, \qquad i = 1,..., m,$$

where the coefficients A_{ijk} are Hermitian symmetric in i and j. If we let $A(\partial/\partial x)$ denote the $m \times m$ Hermitean symmetric matrix whose entries are

$$A\left(\frac{\partial}{\partial x}\right)_{ij} = \sum_{k=1}^{n} A_{ijk}\frac{\partial}{\partial x_k}$$

then we can abbreviate the system in vector notation $\partial u/\partial t = A(\partial/\partial x)u$.

We are interested in L^2 solutions, meaning $\int |u(x, t)|^2\, dx$ is finite (in fact independent of t). Such solutions are easily obtained by Fourier analysis and have the form $u = M(e^{-itA(\xi)})f$ for $f \in L^2$, where $A(\xi)_{ij} = \sum_{k=1}^{n} A_{ijk}\xi_k$ and $e^{-itA(\xi)}$ is the unitary $m \times m$ matrix defined by spectral theory.

Now we will add some hypotheses to simplify the description of $e^{-itA(\xi)}$. *We assume $n > 1$, the non-zero eigenvlues of $A(\xi)$ for $\xi \neq 0$* have constant multiplicities, and the eigenvalue zero has constant multiplicity m_0. Let us write the non-zero real eigenvalues $\rho_1(\xi),\ldots,\rho_\mu(\xi)$, with multiplicities $m_1,\ldots,m_\mu$ (so $m = m_0 + m_1 + \cdots + m_\mu$), where each $\rho_j(\xi)$ is a C^∞ function for $\xi \neq 0$, homogeneous of degree 1. We let $P_j(\xi)$ denote the $m \times m$ orthogonal projection matrix onto the m_j-dimensional eigenspace associated with $\rho_j(\xi)$, and similarly we define $P_0(\xi)$. Then the entries of these matrices are C^∞ functions of ξ, for $\xi \neq 0$, and are homogeneous of degree zero.

We then have

$$e^{-itA(\xi)} = P_0(\xi) + \sum_{j=1}^{\mu} e^{-it\rho_j(\xi)} P_j(\xi)$$

and so $u = u_0 + u_1 + \cdots + u_\mu$, where $u_0 = M(P_0)f$ and $u_j = M(e^{-it\rho_j})\,M(P_j)f$. The time-independent solution u_0 is usually "extraneous" and is ignored or factored out in applications. The terms u_j are waves with phase function ρ_j. In order to obtain the analogues of Theorem 4.1 and 4.2 we need to make one additional assumption: *the Hessian matrix of second derivatives of each ρ_j has rank $n-1$ for each $\xi \neq 0$.* Note that $n-1$ is the maximal rank possible because the homogeneity 1 implies

$$\sum_{i=1}^{n} \xi_i \frac{\partial^2 \rho_j}{\partial \xi_i\, \partial \xi_k} = 0.$$

This can be thought of as a curvature hypothesis, since it is equivalent to the statement that the hyperface $\{\xi : \rho(\xi) = 1\}$ has everywhere positive curvature. This hypothesis appears often in the literature (for instance [2, 3, 7, 11]), but it does not appear to have a name. We propose to call systems which satisfy this hypothesis *well-rounded.* The significance of the condition is indicated by the following:

PROPOSITION 5.1. *Let $\rho(\xi)$ be homogeneous of degree 1 and C^∞ for $\xi \neq 0$. Then the Hessian of ρ has rank $n-1$ for all $\xi \neq 0$ if and only if the mapping $\nabla\rho(\xi)/|\nabla\rho(\xi)|$ restricted to the sphere S^{n-1} is a diffeomorphism onto S^{n-1}.*

Proof. The fact that ρ is homogeneous of degree 1 means

$$\sum \xi_j \frac{\partial \rho}{\partial \xi_j} = \rho$$

so $\nabla\rho$ never vanishes. Thus $\nabla\rho/|\nabla\rho|$ is a well-defined smooth mapping of S^{n-1} to S^{n-1}. Since the radial derivatives is always positive or negative, the mapping $\nabla\rho/|\nabla\rho|$ is homotopic to $\pm$ identity (just phase out the non-radial components). It follows from topological considerations that the mapping is always onto, and it is a diffeomorphism if and only if it is locally a diffeomorphism. The condition for the mapping to be locally a diffeomorphism is that the differential be invertible. Since $\nabla\rho/|\nabla\rho|$ is homogeneous of degree zero when regarded as a mapping of $\mathbb{R}^n\backslash\{0\}$ to $\mathbb{R}^n\backslash\{0\}$, the condition is equivalent to: the rank of

$$\frac{\partial}{\partial\xi_j}\left(\frac{\partial\rho/\partial\xi_k}{|\nabla\rho|}\right)$$

equals $n-1$ at every point $\xi\neq 0$. This is almost what we want, except for the factor $|\nabla\rho|$ in the denominator. However, we know that $\nabla\rho$ is not perpendicular to ξ, and both $\partial\rho/\partial\xi_k$ and $(\partial\rho/\partial\xi_k)/|\nabla\rho|$ have zero derivative in the ξ-direction, so the ranks of both $\partial^2\rho/\partial\xi_k\,\partial\xi_j$ and

$$\frac{\partial}{\partial\xi_j}\left(\frac{\partial\rho/\partial\xi_k}{|\nabla\rho|}\right)$$

are equal to the rank of the restriction to the subspace $(\nabla\rho)^{\perp}$. But if $\eta\in(\nabla\rho)^{\perp}$, then

$$\frac{\partial}{\partial\xi_j}\left(\sum_{k=1}^{N}\eta_k\frac{\partial\rho}{\partial\xi_k}\Big/|\nabla\rho|\right)=\left(\sum_{k=1}^{n}\eta_k\frac{\partial^2\rho}{\partial\xi_j\,\partial\xi_k}\right)\Big/|\nabla\rho|;$$

hence the factor $|\nabla\rho|$ does not enter into the computation of the rank of the matrix. Q.E.D.

In view of the proposition we may define $\tau_j\colon S^{n-1}\to S^{n-1}$ to be the inverse of the mapping $\nabla\rho_j/|\nabla\rho_j|$, and then extend τ_j to $\mathbb{R}^n\backslash\{0\}$ to be homogeneous of degree zero. Thus $\nabla\rho_j(\tau_j(x))$ points in the direction x. Fix $\varepsilon>0$ and define $A_j=\{x\colon x-\nabla\rho_j(\tau_j(x))|<\varepsilon\}$. Note that A_j contains a neighborhood of the range of $\nabla\rho_j$ because if $x=\nabla\rho_j(\xi)$, then $\nabla\rho_j(\xi)/|\nabla\rho_j(\xi)|=x/|x|$ so $\tau_j(x)=\xi/|\xi|$ and $x=\nabla\rho_j(\tau_j(x))$ because $\nabla\rho_j$ is homogeneous of degree zero. By taking ε small enough we can assure that A_j is disjoint from the origin, so it is a "shell." The following result generalizes work of Bardos and Costa [1].

THEOREM 5.2. *If u is any L^2 solution of a well-rounded system, then*

$$\left\|\chi\left(\left|\frac{x}{t}-\nabla\rho_j(\tau_j(x))\right|<\varepsilon\right)u_j-u_j\right\|_2\to 0$$

as $t\to\pm\infty$ for $j=1,\ldots,\mu$.

Proof. We apply Theorem 2.4 to the waves $u_j=M(e^{-it\rho_j})\,M(P_j)f$. Q.E.D.

Theorem 5.2 can also be established by an integration by parts argument similar to the basic estimates concerning wave-front sets of Fourier integral operators [7, p. 44]. However, the integration by parts argument is very delicate and the above proof is considerably simpler.

Next we consider the asymptotic localization of the waves u_j in the range variables (both Theorems 5.2 and 5.3 reduce exactly to Theorems 4.1 and 4.2 when the wave equation is considered as a symmetric hyperbolic system in terms of grad u and the extraneous solutions are ignored).

THEOREM 5.3. *If u is any L^2 solution of a well-rounded system, then* $\|u_j - P_j(\tau_j(\mp x))\, u_j\|_2 \to 0$ *as* $t \to \pm\infty$, *for* $j = 1, 2,..., \mu$.

Proof. We apply Corollary 2.3 to the waves $u_j = M(e^{-it\rho_j}) M(P_j) f$ and $g(x) = P_j(\tau_j(-x))$. Note that g is homogeneous of degree zero so it is continuous except at the origin while the range of $\nabla\rho_j$ has positive distance to the origin. Also g is bounded so the corollary applies. Since g is homogeneous of degree zero, we have $g(x/t) = g(x)$ for $t > 0$. Thus the asymptotic conjugation relation says $\| g(x)\, u_j - M(g(-\nabla\rho_j(\xi))\, u_j\|_2 \to 0$ as $t \to +\infty$. But $g(-\nabla\rho_j(\xi)) = P_j(\tau_j(\nabla\rho_j(\xi)))$ and $P_j(\tau_j(\nabla\rho_j(\xi))) = P_j(\tau_j(\nabla\rho_j(\xi)/|\nabla\rho_j(\xi)|)) = P_j(\xi)$ by the homogeneity and the definition of τ_j. Thus the asymptotic conjugation relation says

$$\|M(P_j(\xi))\, u_j - P_j(\tau_j(-x))\, u_j\|_2 \to 0$$

as $t \to +\infty$. But $P_j(\xi)$ are projection matrices, so $P_j(\xi)\, P_j(\xi) = P_j(\xi)$ and so $M(P_j(\xi))\, u_j = u_j$. This establishes the asymptotic relation for u_j as $t \to +\infty$.

To get the result for $t \to -\infty$ we observe that $g(x/t) = P_j(\tau_j(x))$ for $t < 0$ so the asymptotic conjugation relation says

$$\|P_j(\tau_j(x))\, u_j - \ M(g(\nabla\rho_j(\xi))\, u_j\|_2 \to 0$$

as $t \to -\infty$, and the rest of the proof is the same. Q.E.D.

Remark. In a number of important examples the eigenvalues $\rho_j(\xi)$ occur in $\pm$ pairs. Changing notation slightly we can write the eigenvalues as $\pm\rho_j(\xi)$ with $\rho_j(\xi)$ positive. If $P_j(\xi)$ denotes the projection onto the eigenspace with eigenvalue $+\rho_j(\xi)$, then the relation $A(-\xi) = -A(\xi)$ implies that $P_j(-\xi)$ is the projection onto the eigenspace with eigenvalue $-\rho_j(\xi)$. In this case we can write the solution $u = u_0 + u_1 + \cdots + u_\mu$, where $u_j = u_j^+ + u_j^-$ and $u_j^\pm = M(e^{\mp it\rho_j}) M(P_j(\pm\xi)) f$ for $j \geqslant 1$. Theorems 5.2 and 5.3 then remain true as stated with these changes of notation.

Finally we discuss the relationship between these results and the theory of bicharacteristics. The characteristic polynomial of the system is $\det(\tau I - A(\xi)) = \tau^{m_0} \sum_{j=1}^{\mu} (\tau - \rho_j(\xi))^{m_j}$. Ignoring the extraneous part τ^{m_0}, the characteristics form μ cones $(\rho_j(\xi), \xi)$ in $\mathbb{R}^{n+1}$ as ξ varies in $\mathbb{R}^n \setminus \{0\}$. Corresponding to each point $(\rho_j(\xi), \xi)$ of the characteristic cones—or more precisely each ray, since replacing ξ by a positive multiple does not have any effect—there is a family of bicharacteristic flows through space-time (because the equations are constant coefficient there is no change in the cotangent space) of the form $(x(s), t(s)) = (x_0 + s\nabla\rho_j(\xi), t_0 + s)$. This flow describes the propagation of singularities of the Fourier integral operator $M(e^{-it\rho})$. If we start the flow at the origin $x_0 = 0$, $t_0 = 0$, then after time t elapses it will move to $(t\nabla\rho_j(\xi), t)$. If we set $\xi = \tau_j(x)$ this becomes $(t\nabla\rho_j(\tau_j(x)), t)$, so the image in space of the origin under all the bicharacteristic flows associated with ρ_j after time t elapses is exactly the set $\{x: x - t\nabla\rho_j(\tau_j(x)) = 0\}$. Thus Theorem 5.2 says that asymptotically all of u_j lies in a shell made by slightly thickening this set. The thickening is obviously necessary because when $t = 0$ not all of u_j is localized at the origin. Nevertheless most of u_j lies in a bounded set, and in the asymptotic limit the displacement from the origin can be absorbed in the thickening. Clearly Theorem 5.2 could be improved to allow the shell to thicken at a slower rate.

Note that Theorem 5.2 and the above discussion applies to any wave of the form $M(e^{-it\rho_j})f$. But since we are dealing with only the waves $M(e^{-it\rho_j}) M(P_j)f$ we can say more. To each ray $(\rho_j(\xi), \xi)$ of the characteristic cones we can associate the range of the projection $P_j(\xi)$ (this is the span of the eigenvectors associated with the eigenvalue $\rho_j(\xi)$). Theorem 5.3 says $u_j(x, t)$ asymptotically lies in the subspace associated with the characteristic ray corresponding to the bicharacteristic flow that takes the origin to x in time t. In this sense we may say that asymptotically u_j propagates along the bicharacteristic flows.

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